

THE EFFECT OF BOUNDARY CONDITIONS ON MIXING OF 2D POTTS MODELS AT DISCONTINUOUS PHASE TRANSITIONS

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ABSTRACT. We study the critical stochastic q -state Potts model on the square lattice. Unlike the expected behavior when the phase transition is continuous ($q = 2, 3, 4$)—a universal power-law for the mixing time independently of the boundary conditions—the mixing time at a discontinuous phase transition, t_{MIX} , is highly sensitive to those. It was recently shown by the authors that $t_{\text{MIX}} \geq \exp(cn)$ on an $n \times n$ box with periodic boundary, yet under free or monochromatic boundary conditions $t_{\text{MIX}} \leq \exp(n^{\frac{1}{2}+o(1)})$. In this work we classify this effect under boundary conditions interpolating between these two (torus vs. free/monochromatic) for Swendsen–Wang dynamics at large q . Specifically, we show that alternating boundary conditions, such as red-free-red-free, also induce $t_{\text{MIX}} \geq \exp(cn)$, whereas red-periodic-red-periodic, as well as Dobrushin boundary conditions, such as red-red-free-free, induce sub-exponential mixing.

1. INTRODUCTION

The q -state Potts model at inverse temperature $\beta > 0$ is a generalization of the Ising model ($q = 2$) to $q \geq 3$ possible states. It is a canonical model of statistical physics and is one of the simplest models exhibiting a discontinuous (first-order) phase transition for some choices of q . Concretely, the model on a graph G is a probability distribution over $\{1, \dots, q\}^{V(G)}$ with $\mu(\sigma) \propto \exp(\beta \sum_{ij \in E(G)} \mathbf{1}\{\sigma_i = \sigma_j\})$. Much of the analysis of the Potts model relies heavily on the random cluster (FK) model; the FK model is a model of dependent bond percolation parametrized by (p, q) , identified with the q -state Potts model via the Edwards–Sokal coupling [8] when q is integer and $p = 1 - e^{-\beta}$.

On \mathbb{Z}^2 , substantial recent progress has been made in understanding the Potts and random cluster phase transitions in β and p , respectively. On that geometry, the critical $\beta_c(q)$ was identified [1] for all $q \geq 1$ with the self-dual point $p_{\text{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$. It was shown in [7] that for $q \leq 4$, the phase transition is continuous (there is a unique infinite-volume Gibbs measure at $\beta_c(q)$) whereas for $q > 4$, the phase transition is discontinuous [6] (there are $q + 1$ extremal infinite-volume Gibbs measures corresponding to q ordered phases and a final disordered phase). In the latter case, the phase asymmetry at the critical point is expected to induce order-order and order-disorder surface tensions (known rigorously for q large [14, 19]). In the present work we study the relationship between the order-disorder surface tension and the effect of boundary conditions on mixing times for the critical 2D Potts model.

Specifically, we study the Swendsen–Wang dynamics [23], a non-local Markov chain reversible with respect to μ . Such non-local dynamics has been suggested in the physics literature as a fast MCMC sampler of the Potts model, as it switches between different ordered phases (low-temperature bottlenecks) by moving through the FK representation of the Potts model using the Edwards–Sokal coupling. The authors of [9] analyzed the mixing times of the critical 2D Swendsen–Wang dynamics as the parameter q varied. In [9], polynomial and quasipolynomial upper bounds independent of the boundary conditions were proved for $q \leq 4$; however, when $q > 4$, an exponential lower bound

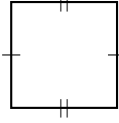
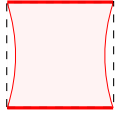


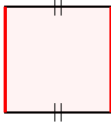
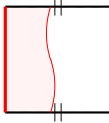
Boundary Conditions	Swendsen–Wang	
Periodic/Alternating		
Dobrushin		
Cylindrical		

FIGURE 1. The boundary conditions considered in Theorems 1–3. Dashed lines indicate free boundary conditions and hash markings $|, ||$ indicate periodic boundary conditions on their respective sides.

on the mixing time held only for the periodic boundary conditions, whereas with free boundary conditions and large q , for instance, the authors proved that $t_{\text{MIX}} \lesssim \exp(n^{o(1)})$.

This sensitivity to boundary conditions is in analogy to the sensitivity of mixing to boundary conditions in the low-temperature stochastic Ising model, where for low enough temperatures, the order-order surface tension is very well understood [5]. There, in two-dimensions, the first sub-exponential bound of $t_{\text{MIX}} \lesssim \exp(n^{\frac{1}{2}+o(1)})$ was obtained in [17] for all sufficiently low temperatures. This was improved to $t_{\text{MIX}} \lesssim \exp(n^{o(1)})$ in [18] and subsequently extended to all $\beta > \beta_c$ and $t_{\text{MIX}} \lesssim n^{O(\log n)}$ in [16]. However, at the *critical* temperature, all known bounds are independent of the boundary conditions; in fact, it is believed that $t_{\text{MIX}} \asymp n^z$ for some universal constant z . Such a picture should hold through $q \leq 4$. (See, e.g., [9, 10] for a more extensive account of related literature.)

When $q > 4$ is sufficiently large, similar cluster expansion techniques lead to an emergent order-disorder surface tension at the critical point, destroying the independence of critical mixing times and boundary conditions. Certain boundary conditions can destabilize the order-disorder phase symmetry, eliminating the exponential bottlenecks in the state space. Therefore, when the boundary conditions are monochromatic or free, the mixing time of the dynamics is proven to be sub-exponential [9] and is believed to have $t_{\text{MIX}} = n^{O(1)}$ with its dynamics governed by mean-curvature flow.

In the present paper, we further investigate the relationship between mixing times and boundary conditions at the critical point of a discontinuous phase transition, classifying mixing times with different natural boundary conditions interpolating between periodic and free/monochromatic. The below results hold for all q for which the presence of order-disorder surface tension is proved and should extend to all $q > 4$.

Theorem 1 (Alternating b.c.). *Let q be large, $\varepsilon > 0$, and let (a_n, b_n, c_n, d_n) be marked vertices on the boundary of $\Lambda_{n,n} = [0, n]^2 \cap \mathbb{Z}^2$ such that they are not all within εn of any one side of $\Lambda_{n,n}$ and are all distance greater than εn from each other. There exists $c(\varepsilon, q) > 0$ such that the critical Swendsen–Wang dynamics on $\Lambda_{n,n}$ with alternating b.c. that are red on the boundary segments (a_n, b_n) and (c_n, d_n) and free elsewhere, has*

$$t_{\text{MIX}} \gtrsim \exp(cn).$$

In particular, this holds with red boundary conditions on $\partial_{\text{E,W}}\Lambda_{n,n}$ and free elsewhere.

Theorem 2 (Dobrushin b.c.). *Let q be large and fix vertices a_n, b_n on $\partial\Lambda_{n,n}$. There exists $c(q) > 0$ such that the critical Swendsen–Wang dynamics on $\Lambda_{n,n}$ with Dobrushin boundary conditions, red on the boundary segment (a_n, b_n) and free elsewhere, has*

$$t_{\text{MIX}} \lesssim \exp(c\sqrt{n \log n}).$$

In particular, this holds with red boundary conditions on $\partial_{\text{S,W}}\Lambda_{n,n}$ and free elsewhere.

Theorem 3 (Cylinders). *Let q be large. The critical Swendsen–Wang dynamics with periodic boundary conditions on $\partial_{\text{N,S}}\Lambda_{n,n}$ and either red or free boundary conditions on each of $\partial_{\text{E}}\Lambda_{n,n}$ and $\partial_{\text{W}}\Lambda_{n,n}$ satisfies*

$$t_{\text{MIX}} \lesssim \exp(n^{1/2+o(1)}).$$

Remark 1.1. Theorems 1–3 all extend also to the Glauber dynamics for the random cluster (FK) model for q sufficiently large (not necessarily integer).

2. PRELIMINARIES

2.1. The q -state Potts model. In this section we formally introduce relevant facts about the Potts and random cluster models (for further details, see, e.g., [12]).

The Potts and FK models. Define the q -state Potts model on a graph $G = (V, E)$ as the probability measure $\mu_{G,\beta,q}$ on $\Omega_{\text{P}} = [q]^V := \{1, \dots, q\}^V$ as

$$\mu_{G,\beta,q}(\sigma) = \mathcal{Z}_{\text{P}}^{-1} e^{\beta \sum_{(i,j) \in E} \mathbf{1}\{\sigma_i = \sigma_j\}},$$

where the normalizer $\mathcal{Z}_{\text{P}}^{-1}$ is the *partition function*. Define the random cluster (FK) model on a graph G as the probability measure $\pi_{G,p,q}$ on state space $\Omega_{\text{RC}} = \{0, 1\}^E$ as

$$\pi_{G,p,q}(\omega) = \mathcal{Z}_{\text{RC}}^{-1} p^{o(\omega)} (1-p)^{E-o(\omega)} q^{k(\omega)},$$

where $o(\omega) = \sum_{e \in E} \omega(e)$ and $k(\omega)$ is the number of connected components (clusters) in the subgraph of G induced by ω (we count singletons as their own clusters). We call edges that have $\omega(e) = 1$ open, or wired, and edges that have $\omega(e) = 0$, closed, or free.

Potts and FK boundary conditions. Consider the Potts and FK models on $G = (V, E)$ with boundary $\partial G \subset G$. A Potts boundary condition on $\partial G \subset G$ is an assignment of spin values $\eta \in [q]^{\partial G}$ so that $\mu_{\beta,q,G}^{\eta} = \mu_{\beta,q,G}(\cdot \mid \sigma|_{\partial G} = \eta)$. An FK boundary condition on ∂G is a partition of $V(\partial G)$ so that we fix a component structure of $V(\partial G)$: e.g., the wired boundary condition is the trivial partition $\{\partial G\}$.

The red Potts boundary condition is an assignment of $\sigma(i) = 1$ to all vertices of ∂G where we call the state “1”, red, or R . The wired FK boundary condition is that

in which all of $V(\partial G)$ is in the same boundary component. The free Potts and FK boundary conditions are identified with the lack of assignments to the vertices of ∂G .

Edwards–Sokal coupling. The Edwards–Sokal coupling [8] is a coupling of the Potts and FK measures on a graph G that enables us to reduce the study of the q -state Potts model, to the study of the FK model at integer q . The joint probability assigned to (σ, ω) , where $\sigma \in \Omega_p$ is a q -state Potts configuration at inverse-temperature $\beta > 0$ and $\omega \in \Omega_{\text{RC}}$ is an FK configuration with parameters $(p = 1 - e^{-\beta}, q)$, is proportional to

$$\prod_{xy \in E(G)} \left[(1-p) \mathbf{1}\{\omega(xy) = 0\} + p \mathbf{1}\{\omega(xy) = 1, \sigma(x) = \sigma(y)\} \right].$$

Planar duality. Throughout this paper we are concerned only with the Potts and FK models on planar graphs, and specifically rectangular subsets $\Lambda_{n,m} \subset \mathbb{Z}^2$ with vertices

$$\Lambda_{n,m} = \llbracket 0, n \rrbracket \times \llbracket 0, m \rrbracket := \{k \in \mathbb{Z} : 0 \leq k \leq n\} \times \{k \in \mathbb{Z} : 0 \leq k \leq m\}$$

and nearest-neighbor edges. For a general subset $D \subset \mathbb{Z}^2$, the boundary ∂D will be the set of vertices in D with neighbors in $\mathbb{Z}^2 - D$, e.g., $\partial \Lambda_{n,m} = \{0, n\} \times \{0, m\}$.

A very useful tool in the study of these models when G is planar is the planar duality of the FK model. For a planar graph G , let G^* be its dual graph. To every FK configuration ω , we associate the dual configuration ω^* given by $\omega^*(e^*) = 1$ if and only if $\omega(e) = 0$ (where e^* is the dual-edge intersecting e). A simple calculation yields that at the self-dual point $p_{\text{sd}}(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$, we have $\pi_G(\omega) = \pi_{G^*}(\omega^*)$.

In the presence of boundary conditions ξ on G , the same holds for the corresponding dual boundary conditions ξ^* on G^* ; these are determined on a case by case basis such that $(\xi^*)^* = \xi$, e.g., the wired and free boundary conditions are dual to each other.

Throughout the paper, for two vertices x, y we will write $x \xleftrightarrow{D} y$ if they are in the same component in $\omega|_D$; when we include an asterisk, we mean x and y are in the same dual-component, i.e. they are in the same component of the dual configuration ω^* .

FKG inequality and monotonicity. When $q \geq 1$, the FK model satisfies positive correlation (FKG) inequalities: if A and B are increasing events in the edge configuration, for all $q \geq 1$, and a boundary condition ξ , we have $\pi_{G,p,q}^\xi(A \cap B) \geq \pi_{G,p,q}^\xi(A) \pi_{G,p,q}^\xi(B)$.

This yields the following monotonicity in boundary conditions: for FK boundary conditions $\xi' \geq \xi$ (the partition associated to ξ is finer than ξ'), $\pi_{G,p,q}^{\xi'}(A) \geq \pi_{G,p,q}^\xi(A)$.

The Potts and FK phase transition. On \mathbb{Z}^2 , the FK and Potts models undergo a phase transition from—in the FK setting—existence a.s. of an infinite cluster at $p > p_c$ to a.s. no infinite cluster at $p < p_c$. In [1] it was proved that for all $q \geq 1$, $p_c(q) = p_{\text{sd}}(q)$. This corresponds to a Potts phase transition from a unique infinite-volume Gibbs measure when $\beta < \beta_c$ to q different extremal Gibbs measures corresponding to weak-limits of boundary conditions of the q colors when $\beta > \beta_c$.

While for $q \leq 4$, the phase transition described above is continuous [7], when $q > 4$, the phase transition is discontinuous [6]: there are two extremal FK Gibbs measures at $p = p_c$ corresponding to the wired and free boundary conditions at infinity, $\pi_{\mathbb{Z}^2}^1 \neq \pi_{\mathbb{Z}^2}^0$ (resp. at $\beta = \beta_c$, $q+1$ extremal Potts Gibbs measures corresponding to the q different

colors, along with a disordered phase with free boundary conditions at infinity). Thus, we have the following: let $q > 4$ and $p = p_c$; there exists $c(q) > 0$ such that

$$\pi_{\mathbb{Z}^2}^0(0 \longleftrightarrow \partial(\llbracket -n, n \rrbracket^2)) \lesssim e^{-cn}. \quad (2.1)$$

Cluster expansion. A key tool for us will be the following cluster expansion technique extensively developed in [5] for the low-temperature Ising model, and later extended to the large q critical Potts/FK model (at the discontinuous phase transition point)—the following estimates are expected to hold at p_c for *all* $q > 4$.

Definition 2.1. Define an FK order-disorder interface, denoted \mathcal{I} in D between boundary points a, b as the outer-most primal crossing adjacent to the dual-component of the connected component of ∂D above a, b . Likewise, a dual FK interface is the outer-most dual-crossing adjacent to the primal component of the connected component of ∂D below a, b . For more general boundary, denote by \mathcal{I} as a compatible collection of such interfaces between all boundary segments that are wired.

Definition 2.2. For an angle $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$, define an *edge-cluster weight function* $\Phi(\mathcal{C}, \mathcal{I})$ as a function with first argument that is a connected set of bonds in \mathcal{S}_n and second argument that is an FK interface connecting $(0, 0)$ to $(n, n \tan \phi)$, satisfying

- (1) $\Phi(\mathcal{C}, \mathcal{I}) = 0$ when $\mathcal{C} \cap \mathcal{I} = \emptyset$,
- (2) $\Phi(\mathcal{C}, \mathcal{I}_1) = \Phi(\mathcal{C}, \mathcal{I}_2)$ when $\Pi_{\mathcal{C}} \cap \mathcal{I}_1 = \Pi_{\mathcal{C}} \cap \mathcal{I}_2$,
- (3) $\Phi(\mathcal{C}, \mathcal{I}) = \Phi(\mathcal{C} + (0, s), \mathcal{I}_1)$ when $\mathcal{I}_1 \cap \Pi_{\mathcal{C}} = (0, s) + \mathcal{I} \cap \Pi_{\mathcal{C}}$,
- (4) $\Phi(\mathcal{C}, \mathcal{I}) \leq \exp(-\lambda d(\mathcal{C}))$.

where $\lambda > 0$, $d(\mathcal{C})$ is the length of the shortest connected subgraph of \mathbb{Z}^2 containing all boundary edges of \mathcal{C} , and $\Pi_{\mathcal{C}} = \{(x, y) \in \mathbb{Z}^2 : \exists y' \text{ s.t. } (x, y') \in \mathcal{C}\}$. Henceforth, we use the specific choice of Φ giving rise to the FK distribution on wired-free interfaces, i.e.,

$$\pi_{\mathcal{S}_n}^{1,0,\phi}(\mathcal{I}) = \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})};$$

the weight function $\Phi(\mathcal{C}, \mathcal{I})$ is given explicitly by Proposition 5 of [19].

Surface tension. The order-disorder surface tension will play a large role in both upper and lower bounds studying the effect of boundary conditions on mixing in the phase coexistence regime. While in the low-temperature regime, there is the presence of an order-order surface tension (see e.g., the $q = 2$ case) that leads to sensitivity of mixing to boundary conditions [17, 18], the disordered phase is unique to the critical point.

Let $\mathcal{S}_n = \llbracket 0, n \rrbracket \times \llbracket -\infty, \infty \rrbracket$, define the *cigar-shaped region* for $d, \kappa > 0$,

$$U_{\kappa,d,\phi} = \mathcal{S}_n \cap \left\{ (x, y) \in \mathbb{Z}^2 : |y - x \tan \phi| \leq d \left| \frac{x(n-x)}{n} \right|^{\frac{1}{2} + \kappa} \right\}, \quad (2.2)$$

and let $(1, 0, \phi)$ boundary conditions on \mathcal{S}_n denote boundary conditions that are wired on $\partial \mathcal{S}_n \cap \{(x, y) \in \mathbb{Z}^2 : y \geq \phi x\}$ and free elsewhere on $\partial \mathcal{S}_n$.

Definition 2.3. The order-disorder surface tension on \mathcal{S}_n in the direction ϕ is defined as $\tau_{1,0}(\phi) = \lim_{n \rightarrow \infty} \frac{1}{\beta_c n^2} [\log(\mathcal{Z}_{\mathcal{S}_n}^{1,0,\phi}) - \log((\mathcal{Z}_{\mathcal{S}_n}^1 \mathcal{Z}_{\mathcal{S}_n}^0)^{1/2})]$.

It was proved first in [14] that for q sufficiently large, at $\beta = \beta_c$ and $\phi \neq \pm \frac{\pi}{2}$, the surface tension $\tau_{1,0}(\phi) > 0$. Extending that following [19], one obtains the following.

Proposition 2.4 ([19, Proposition 5]). *Consider the critical FK model on \mathcal{S}_n and fix a $\delta > 0$. There exists some q_0 such that for all $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$, there exists $c(\delta) > 0$ such that for every $q \geq q_0$ and every $\kappa > 0$,*

$$\pi_{\mathcal{S}_n}^{1,0,\phi}(\mathcal{I} \not\subset (\{(x, y) : y = \lfloor \phi x \rfloor\} + \{0\} \times \llbracket -h, h \rrbracket)) \lesssim n^2 \exp(-ch^2/n).$$

The following is a finer result, that reformulates results of [5] in the FK setting.

Proposition 2.5. *Consider the critical FK model on a domain $\bar{V}_n \supset U_{\kappa,d,\phi}$ and let Φ be the FK order-disorder weight function. Let $\tilde{\Phi}$ be any function satisfying*

$$\tilde{\Phi}(\mathcal{C}, \mathcal{I}) = \Phi(\mathcal{C}, \mathcal{I}) \text{ when } \mathcal{C} \subset U_{\kappa,d,\phi}, \quad \tilde{\Phi}(\mathcal{C}, \mathcal{I}) \leq \exp(-\lambda d(\mathcal{C}))$$

(e.g., $\tilde{\Phi} = \Phi \mathbf{1}_{\{\mathcal{C} \subset U_{\kappa,d,\phi}\}}$). Let $\tilde{\mathcal{Z}}(\bar{V}_n, \phi)$ be the partition function with weights $\tilde{\Phi}$ on \bar{V}_n (see [5, 19]). Then there exist $q_0 > 0$ and $f(\kappa) \lesssim \kappa^{-1}$ such that for all $q \geq q_0$,

$$|\log \tilde{\mathcal{Z}}(\bar{V}_n, \phi) - \log \mathcal{Z}(\mathcal{S}_n, \phi)| \lesssim (\log n)^{f(\kappa)}. \quad (2.3)$$

Moreover, for $\tau_{1,0}(\phi)$ the order-disorder surface tension in the direction of ϕ ,

$$|\log \tilde{\mathcal{Z}}(\bar{V}_n, \phi) - n \log(1 + \sqrt{q})(\cos \phi)^{-1} \tau_{1,0}(\phi)| \lesssim (\log n)^{f(\kappa)}, \quad (2.4)$$

and the large deviation estimate of Proposition 2.4 holds for \bar{V}_n and $d/2$.

2.2. Markov chain mixing times. We introduce the relevant dynamical quantities and techniques in the study of mixing times; for an extensive treatment, see [15].

Mixing times. Consider a Markov chain with finite state space Ω , reversible w.r.t. invariant measure π . Define the total variation distance between measures μ, ν on Ω as

$$\|\mu - \nu\|_{\text{TV}} = \frac{1}{2} \|\mu - \nu\|_{\ell_1} = \sup_{A \subset \Omega} |\mu(A) - \nu(A)|,$$

also defined as a coupling distance $\|\mu - \nu\|_{\text{TV}} = \inf \{\mathbb{P}(X_t \neq Y_t) : X \sim \mu, Y \sim \nu\}$, where the infimum is over all couplings (X, Y) . Then for a discrete time Markov chain with transition kernel $P(\cdot, \cdot)$ (resp., continuous time with heat kernel $H_t(\cdot, \cdot)$), we define the total variation mixing time of the chain as

$$t_{\text{MIX}} = \inf \left\{ t : \max_{x \in \Omega} \|H_t(x, \cdot) - \pi\|_{\text{TV}} < 1/(2e) \right\}.$$

Spectral gap and Dirichlet form. One commonly used technique to bound mixing time of a Markov chain with transition kernel P is to bound the *spectral gap* of P . For a transition matrix P reversible w.r.t., π , by Perron-Frobenius it has largest eigenvalue 1, and by reversibility has real spectrum, which we can enumerate $1 > \lambda_1 \geq \lambda_2 \geq \dots$. Then define the spectral gap, $\text{gap} = 1 - \lambda_1$ satisfying the following relation with t_{MIX} :

$$\text{gap}^{-1} - 1 \leq t_{\text{MIX}} \leq \log\left(\frac{2e}{\pi_{\min}}\right) \text{gap}^{-1}, \quad (2.5)$$

where $\pi_{\min} = \min_{x \in \Omega} \pi(x)$. The variational form of the spectral gap is

$$\text{gap} = \inf_{f: f \neq 0, \mathbb{E}f=0} \frac{\mathcal{E}(f, f)}{\mathbb{E}_\pi(f^2)}$$

where the Dirichlet form $\mathcal{E}(f, f) = \sum_{x, y \in \Omega} \pi(x) P(x, y) (f(y) - f(x))^2$.

Swendsen–Wang dynamics. The main dynamics we consider is the Swendsen–Wang dynamics [23], though some of our results carry over to the Potts Glauber dynamics as well. Swendsen–Wang dynamics for the q -state Potts model on $G = (V, E)$ at inverse temperature β is the following discrete-time Markov chain reversible w.r.t. $\mu_{G,\beta}$. From a spin configuration $\sigma \in \Omega_p$ on G , generate a new state $\sigma' \in \Omega_p$ as follows.

- (1) Introduce auxiliary FK edge variables and set $e = xy \in E$ to be closed with probability 1 if $\sigma_x \neq \sigma_y$ and with probability $e^{-\beta}$ if $\sigma_x = \sigma_y$.
- (2) For every connected component of the edge configuration, reassign the cluster, collectively, an i.i.d. color in $[q]$, to obtain the new configuration σ' .

Heat-bath Glauber dynamics. Continuous-time heat-bath Glauber dynamics for the FK model on G with boundary conditions ξ assigns i.i.d. rate-1 Poisson clocks to the edges: when the clock at $e \in E$ rings, $\omega(e)$ is resampled via $\pi_G^\xi(\omega|_{\{e\}} \mid \omega|_{E-\{e\}} = X_t|_{E-\{e\}})$.

Monotonicity and the grand coupling. The random mapping representation of the FK Glauber dynamic views the updates as a sequence $(J_i, U_i, T_i)_{i \geq 1}$, in which $T_1 < T_2 < \dots$ are the update times, the J_i 's are i.i.d. uniform edges (the updated locations), and the U_i 's are i.i.d. uniform on $[0, 1]$: at time T_i , writing $J_i = xy$, the dynamics replaces the value of $\omega(J_i)$ by $\mathbf{1}\{U_i \leq p\}$ if $x \longleftrightarrow y$ in $\Lambda - \{J_i\}$ and by $\mathbf{1}\{U_i \leq \frac{p}{p+q(1-p)}\}$ otherwise.

The heat-bath Glauber dynamics for the FK model at $q \geq 1$ is *monotone*: for every two FK configurations $\omega_1 \geq \omega_2$ and every $t \geq 0$, we have $H_t(\omega_1, \cdot) \succeq H_t(\omega_2, \cdot)$.

The *grand coupling* for Glauber dynamics is a coupling of the chains from all initial configurations on Λ which, via the above random mapping representation, uses the same update sequence $(J_i, U_i, T_i)_{i \geq 1}$ for each one of these chains. For $q \geq 1$, this coupling preserves the partial ordering of the configurations at all times $t \geq 0$.

Block dynamics. A key ingredient in many upper bounds on mixing is the block dynamics technique due to Martinelli (see [17, §3]) for bounding the spectral gap of the Glauber dynamics. Suppose B_1, \dots, B_k is an arbitrary cover of G . Then the block dynamics is the corresponding Glauber dynamics that updates one block (instead of one site) at a time: each block is assigned a rate-1 Poisson clock; when the clock at B_i rings, resample the configuration on B_i according to $\pi_{B_i}^\sigma$ where the boundary conditions σ are given by those induced by the chain restricted to $G - B_i$.

The following was originally stated for spin systems with finite range interactions in [17], and extended to the more general setup we present in [9, Theorem 2.9].

Theorem 2.6 ([17, Proposition 3.4]). *Consider a continuous-time single site Markov chain for a spin system on G with boundary condition ζ and transition rates $c_G^\zeta(\sigma, \sigma')$, which is reversible w.r.t. a Gibbs distribution μ_G^ζ that has the Domain Markov property. Let gap_G^ζ be the spectral gap of the single-site dynamics on G and gap_B^ζ be the spectral gap of the block dynamics corresponding to B_1, \dots, B_k , an arbitrary cover of G . Let*

$$\chi = \sup_{x \in G} \#\{i : B_i \ni x\}, \quad \text{and} \quad M = \sup_{i, \varphi} \sup_{\sigma, \sigma' : c_G^\zeta(\sigma, \sigma') > 0} \frac{c_{B_i}^\varphi(\sigma|_{B_i}, \sigma'|_{B_i})}{c_G^\zeta(\sigma, \sigma')}.$$

Then we have $\text{gap}_G^\zeta \geq (\chi M)^{-1} \text{gap}_B^\zeta \inf_{i, \varphi} \text{gap}_{B_i}^\varphi$.

Comparison between cluster and Glauber dynamics. Key to our results will be the following comparison estimates. They allow us to reduce the analysis of the Swendsen–Wang dynamics on Λ to the analysis of the FK Glauber dynamics.

Theorem 2.7 ([21, 22]). *Let $q \geq 2$ be integer. Let gap_{P} and gap_{RC} be the spectral gaps of Glauber dynamics for Potts and FK model on a graph G on m edges and maximum degree Δ , resp., and let gap_{SW} be the spectral gap of Swendsen–Wang. Then*

$$\text{gap}_{\text{P}} \leq 2q^2(qe^{2\beta})^{4\Delta} \text{gap}_{\text{SW}}, \quad (2.6)$$

$$\text{gap}_{\text{RC}} \leq \text{gap}_{\text{SW}} \leq 16\text{gap}_{\text{RC}} m \log m. \quad (2.7)$$

Potts boundary conditions and mixing. The comparison estimates of the previous section are only valid in the absence of boundary conditions while the focus of this paper is the influence of various fixed boundary conditions.

In particular, the estimates of Theorem 2.7 hold immediately for Λ with periodic or free boundary conditions. For other boundary conditions, deform Λ as in the following remark; this could, however, distort the maximum degree Δ exponentially in n in (2.6).

Remark 2.8. For any FK boundary condition ξ on G , we can define a (not necessarily planar) graph \tilde{G} by identifying all vertices of every boundary component of ξ with a single vertex in \tilde{G} , and keeping the same edge structure. Then the FK Glauber dynamics on \tilde{G} with free boundary conditions is the same as that on G with boundary conditions ξ . In such a case, let $\partial\tilde{G}$ denote the set of vertices in \tilde{G} that arise from the boundary components of G .

Remark 2.9. The dependence on Δ in Theorem 2.7 can be improved to exponential in the maximum degree of all but one vertex (see [21, Theorem 1’]), whence Δ in Eq. (2.6) can be replaced with the second largest degree of G .

The following is a consequence of spin symmetry of the Swendsen–Wang dynamics.

Fact 2.10. *Consider the Swendsen–Wang dynamics on G with fixed boundary conditions η , which we identify with a graph \tilde{G} given by Remark 2.8, identifying all boundary vertices of each color with a single vertex. If $\partial\tilde{G}$ consists of at most one vertex, then*

$$\text{gap}_{\eta, G}^{-1} = \text{gap}_{\eta, \tilde{G}}^{-1} = \text{gap}_{0, \tilde{G}}^{-1},$$

where (η, G) is the graph G with Potts boundary conditions η , (η, \tilde{G}) assigns the vertices of $\partial\tilde{G}$ those given by η and $(0, \tilde{G})$ is the Potts model on \tilde{G} with no boundary conditions.

Remark 2.9 and Fact 2.10 imply that for $\Lambda_{n,n}$ with FK boundary conditions ξ with at most one nontrivial boundary component, corresponding to Potts boundary conditions η (an assignment of red to the nontrivial boundary components of ξ),

$$t_{\text{MIX}}^{\text{RC}} \lesssim t_{\text{MIX}}^{\text{SW}} \lesssim n^2 t_{\text{MIX}}^{\text{RC}}.$$

3. SUB-EXPONENTIAL MIXING WITH SYMMETRY-BREAKING BOUNDARY

In this section we prove sub-exponential upper bounds on the inverse spectral gap of Swendsen–Wang dynamics for general Dobrushin boundary conditions. Here, there is a single high probability minimizer of the surface tension for the model, breaking the order-disorder phase symmetry that induces slow mixing in Section 4.

3.1. Surface tension estimates on subsets of $\Lambda_{n,n}$. In this section, we extend the surface tension estimate of Proposition 2.4 to tilted half-infinite and finite strips.

Surface tension estimates on strips. We first define subsets of $\Lambda_{n,n}$ we consider in obtaining the desired sub-exponential upper bounds of Theorem 1–2. Recall the notation

$$\mathcal{S}_n = \llbracket 0, n \rrbracket \times \llbracket -\infty, \infty \rrbracket.$$

Definition 3.1. Define a tilted strip $\mathcal{S}_{b,h,\phi}$ as (for $b \in \mathbb{R}, h \in \mathbb{R}_+, \phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$),

$$\mathcal{S}_{b,h,\phi} = \{(x, y) \in \mathbb{Z}^2 : y \in \llbracket b - h + \phi x, b + h + \phi x \rrbracket\}.$$

Also, define the half-infinite strips

$$\mathcal{H}_{b,\phi}^+ = \{(x, y) \in \mathbb{Z}^2 : y \geq b + \phi x\}, \quad \mathcal{H}_{b,\phi}^- = \{(x, y) \in \mathbb{Z}^2 : y \leq b + \phi x\}.$$

We next bound the probability that the order-disorder interface climbs a vertical height of a above $\partial_s \mathcal{H}_{b,\phi}^+$. The estimate—extending an estimate of [19], which in turn adapts [5] to the FK model, to half-infinite strips—is a consequence of monotonicity of the FK model and Proposition 2.4.

Proposition 3.2. *Let q be large, $b \in \mathbb{R}$ and $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$ for $\delta > 0$, and consider the critical FK model on $\mathcal{H}_{b,\phi}^+$ with boundary conditions $(1, 0, b, \phi)$ denoting wired boundary conditions on $\{(x, y) \in \mathcal{H}_{b,\phi}^+ : y \geq b + \phi x - 1\}$ and free boundary conditions elsewhere on $\partial \mathcal{H}_{b,\phi}^+$. There exist constants $A > 0$ and $c(\delta, q) > 0$ such that the order-disorder interface \mathcal{I} , satisfies*

$$\pi_{\mathcal{H}_{b,\phi}^+}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+a,\phi}^- \cap \mathcal{H}_{b,\phi}^+) \leq A n^2 \exp(-c a^2/n).$$

Proof. The proposition was proven in the case $\phi = 0$ in Proposition 4.2 of [9] combining monotonicity of the FK model in boundary conditions with Proposition 2.4. The same proof carries over to the case $\phi \neq 0$ as long as ϕ is uniformly bounded away from $\pm \frac{\pi}{2}$ as the surface tension estimate of Proposition 2.4 on \mathcal{S}_n is expressed in that setup. ■

The following is the main equilibrium estimate we will use in the proofs of sub-exponential mixing for general Dobrushin boundary conditions.

Proposition 3.3. *Let q be large, $b \in \mathbb{R}$, $h \leq m \leq n$, and $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$ for $\delta > 0$, and consider the critical FK model on $\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}$ with $(1, 0, b, \phi)$ boundary conditions denoting wired on $\partial \mathcal{S}_{b,m,\phi} \cap \mathcal{H}_{b,\phi}^+ \cap \Lambda_{n,n}$ and free elsewhere on $\partial(\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n})$. Then there exists $c(\delta, q) > 0$ such that*

$$\pi_{\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{S}_{b,h,\phi} \cap \Lambda_{n,n}) \lesssim \exp(-c h^2/n).$$

We need the following preliminary estimate.

Lemma 3.4. *Let q be large, $b \in \mathbb{R}$ and $\phi \in [-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$ for $\delta > 0$, and consider the critical FK model on $\Lambda_{n,n}$ on $\mathcal{H}_{b,\phi}^+ \cap \Lambda_{n,n}$ with $(1, 0, b, \phi)$ boundary conditions denoting wired on $\partial(\mathcal{H}_{b,\phi}^+ \cap \Lambda_{n,n}) \cap \mathcal{H}_{b+1,\phi}^+$ and free elsewhere. There exists $c(\delta, q) > 0$ such that*

$$\pi_{\mathcal{H}_{b,\phi}^+ \cap \Lambda_{n,n}}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+a,\phi}^- \cap \mathcal{H}_{b,\phi}^+ \cap \Lambda_{n,n}) \lesssim e^{-c a^2/n}.$$

Proof. Let $B = \mathcal{H}_{b+a,\phi}^- \cap \mathcal{H}_{b,\phi}^+ \cap \Lambda_{n,n}$ and consider the strip $S' = \mathcal{H}_{b,\phi}^+ \cap \mathcal{H}_{n,0}^-$.

For a general domain $D \subset \mathcal{S}_n$ let the boundary conditions $(1, 0, b, \phi)$ on it denote wired on $\partial D \cap \mathcal{H}_{b+1,\phi}^+$ and free elsewhere on ∂D . By monotonicity in boundary conditions, and then inclusion of events,

$$\begin{aligned} \pi_{\mathcal{H}_{b,\phi}^+ \cap \Lambda_{n,n}}^{1,0,b,\phi}(\mathcal{I} \not\subset B) &\leq \pi_{S'}^{1,0,b,\phi}(\mathcal{I} \not\subset \Lambda_{n,n} \cap B) \\ &\leq \pi_{S'}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+a,\phi}^- \cap \mathcal{H}_{b,\phi}^+). \end{aligned}$$

By monotonicity in boundary conditions again,

$$\pi_{S'}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+a,\phi}^- \cap \mathcal{H}_{b,\phi}^+) \leq \pi_{\mathcal{H}_{b,\phi}^+}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+a,\phi}^- \cap \mathcal{H}_{b,\phi}^+).$$

By Proposition 3.2, there exists a $c(\delta, q) > 0$ such that

$$\pi_{\mathcal{H}_{b,\phi}^+}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+a,\phi}^- \cap \mathcal{H}_{b,\phi}^+) \lesssim n^2 e^{-ca^2/n}. \quad \blacksquare$$

Proof of Proposition 3.3. As before, let $(1, 0, b, \phi)$ boundary conditions on $D \subset \mathbb{Z}^2$ denote those that are wired on $\partial D \cap \{y \geq b + \phi x\}$ and free on $\partial D \cap \{y < b + \phi x\}$. By a union bound, write

$$\pi_{\mathcal{S}_{b,m,\phi} \cap \Lambda}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{S}_{b,h,\phi}) \leq \pi_{\mathcal{S}_{b,m,\phi} \cap \Lambda}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b-h,\phi}^+) + \pi_{\mathcal{S}_{b,m,\phi} \cap \Lambda}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+h,\phi}^-) \quad (3.1)$$

and consider the two quantities independently. Observe that the first event on the right-hand side is an increasing event, while the second event is a decreasing event. (By reflection symmetry and self-duality, if we prove the desired bound on the latter, for general b, h, ϕ , it implies the former also.) By monotonicity in boundary conditions,

$$\begin{aligned} \pi_{\mathcal{S}_{b,m,\phi} \cap \Lambda_{n,n}}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+h,\phi}^-) &\leq \pi_{\mathcal{S}_{b,m,\phi} \cap \mathcal{H}_{b-1,\phi}^+ \cap \Lambda_{n,n}}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+h,\phi}^-) \\ &\leq \pi_{\mathcal{H}_{b-1,\phi}^+ \cap \Lambda_{n,n}}^{1,0,b,\phi}(\mathcal{I} \not\subset \mathcal{H}_{b+h,\phi}^-). \end{aligned}$$

The right-hand side above is exactly the probability bounded in Lemma 3.4, from which, along with the symmetry noted above and (3.1), the desired upper bound follows. \blacksquare

We will also need the following bound in order to prove the mixing time upper bounds on cylinders in §3.3. It is an adaptation of the proof of [18, Lemma A.6] from the Ising model to the FK model via Propositions 2.4–2.5.

Proposition 3.5. *Fix q to be large enough and $\varepsilon > 0$, and consider the critical FK model on $\Lambda_{n,h}$ for $n^{\frac{1}{2}+\varepsilon} \leq h \leq n$ with $1, 0$ boundary conditions denoting wired on $\partial_s \Lambda_{n,h}$ and free elsewhere. For $\rho \in (0, 1)$ and δ small enough, there exists $c(\varepsilon) > 0$ such that*

$$\pi_{\Lambda_{n,h}}^{1,0}(\mathcal{I} \cap [\lfloor \frac{n}{2} \rfloor - \delta n, \lfloor \frac{n}{2} \rfloor + \delta n] \times [0, \rho h]) = \emptyset \gtrsim e^{-c(\rho h)^2/n}.$$

Proof. Denote by $B = [\lfloor \frac{n}{2} \rfloor - \delta n, \lfloor \frac{n}{2} \rfloor + \delta n] \times [0, \rho h]$. Recall the definition of the cigar-shaped region $U_{\kappa,d,\phi}$ in (2.2). Following [18], for every $-\log_2 n + 2 \leq i \leq \log_2 n - 2$, let

z_i be the nearest vertex to

$$\left(\tilde{x}_i, d \left| \frac{\tilde{x}_i(n - \tilde{x}_i)}{n} \right|^{\frac{1}{2} + \kappa} \right) \quad \text{where} \quad \tilde{x}_i = \frac{n}{2} + \frac{\operatorname{sgn} i}{4} \sum_{j=1}^{|i|-1} 2^{-j}.$$

Let $U_{z_i, z_{i+1}}$ be the cigar shaped region $z_i + U_{\varepsilon/2, (1-\rho) \wedge \rho, \phi_{z_i, z_{i+1}}}$ where $d = (1 + \rho)h/n^{\frac{1}{2} + \varepsilon}$ and $\phi_{z_i, z_{i+1}}$ is the angle of the vector from z_i to z_{i+1} .

By monotonicity in boundary conditions and $\{\mathcal{I} \cap B = \emptyset\}$ being an increasing event,

$$\pi_{\Lambda_{n,h}}^{1,0}(\mathcal{I} \cup B = \emptyset) \geq \pi_{\mathcal{H}_{h,0}^-}^{1,0}(\mathcal{I} \cap B = \emptyset),$$

where $(1, 0)$ boundary conditions on $\partial\mathcal{H}_{h,0}^-$ are wired on $\partial\mathcal{H}_{h,0}^- \cap \mathcal{H}_{0,0}^-$ and free elsewhere. Since $U_{z_i, z_{i+1}} \cap B = \emptyset$ for all i , if $\mathcal{I} \subset \bigcup_{|i| \leq \log_2 n} U_i$, then the desired property holds.

Lemma A.6 of [18] gives a lower bound on the partition function restricted to interfaces contained in $\bigcup U_i$ as defined above in the setting of the Ising model; the same proof extends that lower bound to the partition function of such FK interfaces, noting that Proposition 2.5 is an analogue of [5, Theorem 4.16]: there exists $c > 0$ such that

$$\sum_{\mathcal{I} \subset \mathcal{H}_{h,0}^-} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})} \mathbf{1}\{\mathcal{I} \subset \bigcup U_i\} \geq \exp(-\beta_c \tau_{1,0}(0)n - c(\rho h)^2/n),$$

where an error of $(\log n)^{O(1)}$ was absorbed into the term $c(\rho h)^2/n$ via the assumption that $h \geq n^{1/2+\varepsilon}$ and a choice of a suitable constant $c > 0$.

Futhermore, Proposition 2.5, in particular (2.4), implies there exists $c > 0$ so that

$$\sum_{\mathcal{I} \subset \mathcal{H}_{h,0}^-} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})} \leq \exp(-\beta_c \tau_{1,0}(0)n + c(\log n)^c).$$

Then writing the desired probability as in [18, (A.36)] as

$$\begin{aligned} \pi_{\mathcal{H}_{h,0}^-}^{1,0}(\mathcal{I} \cap B \neq \emptyset) &\geq \pi_{\mathcal{H}_{h,0}^-}^{1,0}\left(\mathcal{I} \subset \bigcup_{|i| \leq \log_2 n-2} U_i\right) \\ &= \frac{\sum_{\mathcal{I} \subset \mathcal{H}_{h,0}^-} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})} \mathbf{1}\{\mathcal{I} \subset \bigcup U_i\}}{\sum_{\mathcal{I} \subset \mathcal{H}_{h,0}^-} \lambda^{|\mathcal{I}| + \sum_{\mathcal{C} \cap \mathcal{I} \neq \emptyset} \Phi(\mathcal{C}, \mathcal{I})}}, \end{aligned}$$

and plugging in the above lower bound on the numerator and upper bound on the denominator, concludes the proof. \blacksquare

3.2. Dobrushin boundary conditions. In this section, we consider boundary conditions that have free boundary conditions on a subset of $\partial\Lambda$ and red boundary conditions elsewhere. While a telling example in §4 demonstrates that such boundary conditions can induce slow mixing by respecting the order-disorder phase symmetry, the analysis in this section will be restricted to boundary conditions with a single order-disorder phase boundary, thus inducing sub-exponential mixing.

Define a general class of *order-disorder Dobrushin boundary conditions*, whose FK representation is wired on one boundary segment and free elsewhere. Let a_n, b_n be two distinct points on $\Lambda_{n,n}$. Denote by (a_n, b_n) boundary conditions those that are free on the clockwise (starting from the origin) arc (b_n, a_n) and red on (a_n, b_n) .

Proof of Theorem 2. If a_n, b_n are on the same side of $\Lambda_{n,n}$, the desired bound follows by repeating the proof of [9, Eq. (1.3) of Theorem 3, §4.3] exactly; the difference between wired boundary conditions on one side of $\Lambda_{n,n}$ and $a_n = (a_n^1, a_n^2)$, $b_n = (b_n^1, b_n^2)$ being on the same side of $\partial\Lambda_{n,n}$ only manifests in the last step of the block dynamics, where the identity coupling still couples all FK chains with probability 1.

Now suppose that a_n, b_n are not on the same side of $\Lambda_{n,n}$ and let $\phi_n = \tan^{-1}(\frac{a_n^2 - b_n^2}{a_n^1 - b_n^1})$. By symmetry and self-duality, it suffices to consider the case when $a_n \in \partial_w \Lambda_{n,n}$ and $\phi_n \in [-\frac{\pi}{4}, \frac{\pi}{4}]$. Fix any such choice of $a_n, b_n \in \partial\Lambda_{n,n}$ and let $\phi = \phi_n$.

We establish the theorem for Glauber dynamics for the FK model on $\Lambda_{n,n}$. Let $c_1, c_2 > 0$ be the constants obtained from Lemmas 3.2 and (3.2) below respectively. Following the argument of [17, §3] and [9, §4.3], consider block dynamics \mathcal{B} with blocks

$$B_i := \mathcal{S}_{b+i\ell, \ell, \phi} \cap \Lambda_{n,n} \quad (i = -N, \dots, N) \quad \text{for } N = \left\lceil \frac{n}{\ell} \right\rceil - 1,$$

where we choose $\ell = c_3 \sqrt{n \log n}$ for $c_3 = 5/\sqrt{c_1}$, with $c_1(q)$ as given by Proposition 3.2, so $N \sim c_3^{-1} \sqrt{n/\log n}$. Because of our choice of B , as many as N of the B_i will be empty, and henceforth if B_i is empty, we say any associated events hold trivially. The following simple canonical paths estimate (see [3, 4, 13, 20]) crudely bounds the mixing time on subsets of $\Lambda_{n,n}$: for any strip $S = \mathcal{S}_{b,h,\phi} \subset \Lambda_{n,n}$ of height $h \leq n$, and any FK boundary conditions ξ on ∂S , there exists $c > 0$ such that

$$t_{\text{MIX}} \lesssim \exp(ch \log q). \quad (3.2)$$

Indeed this follows from a simple modification of the proof of [9, Lemma 4.1].

By Theorem 2.6 and Eq (3.2), the FK Glauber dynamics has

$$\text{gap}_\Lambda^{-1} \lesssim 2e^{(c_2 \log q)\sqrt{n \log n}} \text{gap}_\mathcal{B}^{-1}, \quad (3.3)$$

and it remains to obtain a corresponding upper bound on $\text{gap}_\mathcal{B}^{-1}$.

Let U_N denote the event that the updates of the block dynamics contain the ordered sequence $(B_{-N}, B_N, \dots, B_{-1}, B_1, B_0)$ (with no other block updates in between) by time

$$T := (2N+1)^{2N+1} = \exp\left(\left(\frac{1}{2c_3} + o(1)\right)\sqrt{n \log n}\right),$$

which satisfies (see e.g., (4.8) of [9])

$$\mathbb{P}(U_N^c) \lesssim e^{-\frac{T}{2}} + e^{-\frac{(2N+1)^{2N}}{2(2N+1)!}} = e^{-e^{(2-o(1))N}} = o(1),$$

where $\text{Po}(T)$ is a mean T Poisson random variable.

We now bound the coupling distance of the block dynamics at time T given U_N . Consider the discrete-time Markov chain $(X_i)_{i=1}^{2N+1}$ that performs the ordered block update sequence $(B_{-N}, B_N, \dots, B_{-1}, B_1, B_0)$, and denote by X_i^0 and X_i^1 two copies of this chain with initial configurations $X_0^0 = 0$ and $X_0^1 = 1$ respectively, coupled via the grand coupling (see §2.2). For each $i = 1, \dots, N$, let

$$A_i = \left\{ X_{2i}^0 \upharpoonright_{E(R_i)} \neq X_{2i}^1 \upharpoonright_{E(R_i)} \right\}, \quad \text{where } R_i := R_i^+ \cup R_i^- \quad \text{with } R_i^\pm = \bigcup_{j=0}^{i-1} B_{\pm(N-j)}.$$

Since $R_i \subset R_{i+1}$ for all i , and $\Lambda - R_N \subset B_0$, by the update order of (X_i) , if it is the case that $X_{2N}^0 \upharpoonright_{R_N} = X_{2N}^1 \upharpoonright_{R_N}$, the identity coupling along with the final block update will yield $X_{2N+1}^0 = X_{2N+1}^1$ on all of $\Lambda_{n,n}$. Hence, we may write

$$\mathbb{P}(X_{2N+1}^0 \neq X_{2N+1}^1) = \mathbb{P}\left(\bigcup_{i=1}^{N-1} A_i\right) \leq \mathbb{P}(A_1) + \sum_{i=2}^N \mathbb{P}(A_{i-1}^c \cap A_i).$$

Observe that the event A_{i-1}^c implies $X_{2(i-1)}^0, X_{2(i-1)}^1$ agree on the boundary conditions $\partial_N B_i, \partial_S B_{-i}$. For positive i , the boundary conditions on $\partial_N B_i$ dominate the measure on such boundary conditions induced by $\pi_{R_i^+ \cup B_i}^0$ and are dominated by that induced by $\pi_{R_i^+ \cup B_i}^{0,1}$, with $(0,1)$ denoting wired on $\partial(R_i^+ \cup B_i) \cap \mathcal{H}_{b+i+1,\phi}^-$ and free elsewhere on $\partial(R_i^+ \cup B_i)$. Thus, for all $i = 1, \dots, N$, by self-duality and monotonicity,

$$\begin{aligned} \mathbb{P}(A_{i-1}^c \cap A_i) &\leq \sum_{e \in E(R_i^+ \cap B_i)} \pi_{R_i^+ \cup B_i}^{0,1}(e \in \omega) - \pi_{R_i^+ \cup B_i}^0(e \in \omega) \\ &\quad + \sum_{e \in E(R_i^- \cap B_{-i})} \pi_{R_i^- \cup B_{-i}}^1(e \in \omega) - \pi_{R_i^- \cup B_{-i}}^{1,0}(e \in \omega), \end{aligned}$$

(and similarly for $\mathbb{P}(A_1)$). By symmetry, it suffices to bound the first sum on the right-hand side uniformly in b and $\text{sgn}(\phi)$. For each $i = 1, \dots, N$, and every $e \in E(R_i^+ \cap B_i)$,

$$\pi_{R_i^+ \cup B_i}^{0,1}(e \in \omega) - \pi_{R_i^+ \cup B_i}^0(e \in \omega) \leq \pi_{R_i^+ \cup B_i}^{0,1}(\mathcal{I} \not\subset B_i - R_i^+).$$

(If the interface does not exceed $\partial_N(B_i - R_i^+)$ under $\pi_{R_i^+ \cup B_i}^{0,1}$, then its immediately adjacent dual-crossing is also open under $\pi_{R_i^+ \cup B_i}^0$ via the monotone coupling, so we can couple the two configurations above that horizontal dual-crossing, including, in particular, all of $R_i^+ \cap B_i$ and thus $\partial_N B_{i-1}$.)

Observe that because the boundary conditions are side-homogenous, conditioning on the configuration below the interface, while revealing the interface, cannot affect the boundary conditions above it because on each side of $\partial R_i^+ \cup B_i$ the boundary conditions are 1/0 and additional connections cannot be induced (cf. the boundary bridges of [10]).

In that case, Lemma 3.4 (noting that the estimate there was uniform in b) with the choice of $a = \frac{1}{2}c_3\sqrt{n \log n}$, and a union bound over all edges, both $R_i^+ \cap B_i$ and $R_i^- \cap B_{-i}$, and all $i = 1, \dots, N$, together imply that, for sufficiently large n ,

$$\mathbb{P}(X_{2N+1}^0 \neq X_{2N+1}^1) \leq 16An^4 \exp(-\frac{1}{4}c_2c_3^2 \log n) = o(1),$$

resulting from the choice of $c_3 = 5/\sqrt{c_2}$. Combined with the probability of the desired ordered sequence appearing in the Poisson clock rings by time T implies that $\text{gap}_{\mathcal{B}}^{-1} \leq T$ for large enough n . Along with (3.3), this completes the proof for the case of Glauber dynamics for the FK model. The comparison estimates (2.6)–(2.7) along with Fact 2.10 imply the analogous bounds for the critical Swendsen–Wang dynamics. \blacksquare

3.3. Sub-exponential mixing on cylinders. For the rectangle $\Lambda_{n,n}$ define boundary conditions (p, R) (resp. $(p, 0)$ or $(p, R, 0)$ boundary conditions) to be periodic boundary conditions on $\partial_{N,S}\Lambda_{n,n}$ and red on $\partial_{E,W}\Lambda_{n,n}$ (resp. free on $\partial_{E,W}\Lambda_{n,n}$ or red on $\partial_W\Lambda_{n,n}$ and free on $\partial_E\Lambda_{n,n}$). We prove the mixing time upper bounds on cylinders with the above boundary conditions (Theorem 3) at the same time. In what follows, we use $C, c > 0$ to denote the existence of a constant (possibly depending on q), where different appearances of C, c at different places may refer to different values.

Definition 3.6 (“wired” / “free” b.c.). Define the class of “wired” boundary conditions as the distribution over boundary conditions on $\Delta \subset \partial\Lambda_{n,m}$ according to $\pi_{\mathbb{Z}^2}^1$ by sampling a configuration on $\mathbb{Z}^2 - \Lambda_{n,m}$ and identifying it with the partition of Δ it induces. Likewise define the class of “free” boundary conditions with respect to $\pi_{\mathbb{Z}^2}^0$.

Proof of Theorem 3. The proof modifies the proof of [9, Eq. (1.4) of Theorem 3]. As before, it suffices to prove the upper bound for the FK Glauber dynamics with corresponding boundary conditions. Fix $\varepsilon, \delta > 0$ small; consider block dynamics with

$$\left\{ \begin{array}{l} B_{2i-1} = \llbracket (i-1)\ell, (i+1)\ell \rrbracket \times \llbracket \delta n, (1-\delta)n \rrbracket \\ B_{2i} = \llbracket (i-1)\ell, (i+1)\ell \rrbracket \times \llbracket 0, \lfloor \frac{n}{2} \rfloor - \delta n \rrbracket \cup \llbracket \lfloor \frac{n}{2} \rfloor + \delta n, n \rrbracket \end{array} \right\} \quad \text{where } \ell = n^{\frac{1}{2}+\varepsilon},$$

for $i = 1, \dots, N$ where $N = \lceil \frac{n}{\ell} \rceil - 1$ and $B_{2N+1} = \Lambda - \cup_{i=1}^N B_{2i-1} \cup B_{2i}$. Since the boundary conditions on $\partial_{N,S}\Lambda_{n,n}$ are periodic, each B_{2i} can be viewed as a single connected rectangle with boundary $\partial_{N,S}B_{2i} = \llbracket (i-1)\ell, (i+1)\ell \rrbracket \times \{\lfloor \frac{n}{2} \rfloor - \delta n, \lfloor \frac{n}{2} \rfloor + \delta n\}$. We prove the mixing time upper bound for $(p, 1, 0)$ boundary conditions whence $(p, 1)$ boundary conditions are the same up to the last deterministic step of the block dynamics, and $(p, 0)$ can be treated by the dual version of the argument we present.

We bound the inverse gap of the sub-blocks as before using Eq. (3.2), so that it remains to lower bound the gap of the block dynamics. By the same computation as in the proof of Theorem 2, the probability of having an ordered sequence $(1, \dots, 2N+1)$ in the clock rings of the blocks B_i in time T is $1 - o(1)$ for $T = (2N+1)^{2N+1}$, so suppose there is such an ordered sequence in time T . We bound the probability of not coupling two discrete-time chains (X_t^1, X_t^0) along this ordered sequence started from the all-wired and all-free initial configurations, via the monotone coupling. Let

$$A_{2i} = \left\{ X_{2i}^0 \upharpoonright_{R_{2i}-B_{2i+1}-B_{2i+2}} \neq X_{2i}^1 \upharpoonright_{R_{2i}-B_{2i+1}-B_{2i+2}} \right\} \quad \text{where} \quad R_i = \bigcup_{j \leq i} B_j.$$

If $X_{2N}^1 = X_{2N}^0$ on $\cup_{i=1}^N B_{2i-1} \cup B_{2i}$, then in the last step of the ordered sequence updates, deterministically $X_{2N}^1 = X_{2N}^0$ on all of Λ . Thus, by definition of A_{2i} , write

$$\mathbb{P}(X_{2N+1}^0 = X_{2N+1}^1) \geq \mathbb{P}(A_2^c) \prod_{i=2}^N \mathbb{P}(A_{2i}^c \mid A_{2i-2}^c). \quad (3.4)$$

We claim that it suffices to lower bound the above probability by $e^{-cn^{\frac{1}{2}+\varepsilon}}$ for some $c(\delta) > 0$ as the probability of not coupling any two block dynamics chains under the monotone coupling in time $Ce^{cn^{\frac{1}{2}+\varepsilon}}T$ would then be less than $\frac{1}{2e}$ for C sufficiently large. In that case, we would have $\text{gap}_B^{-1} \leq CT e^{cn^{\frac{1}{2}+\varepsilon}}$ which combined with the earlier choice

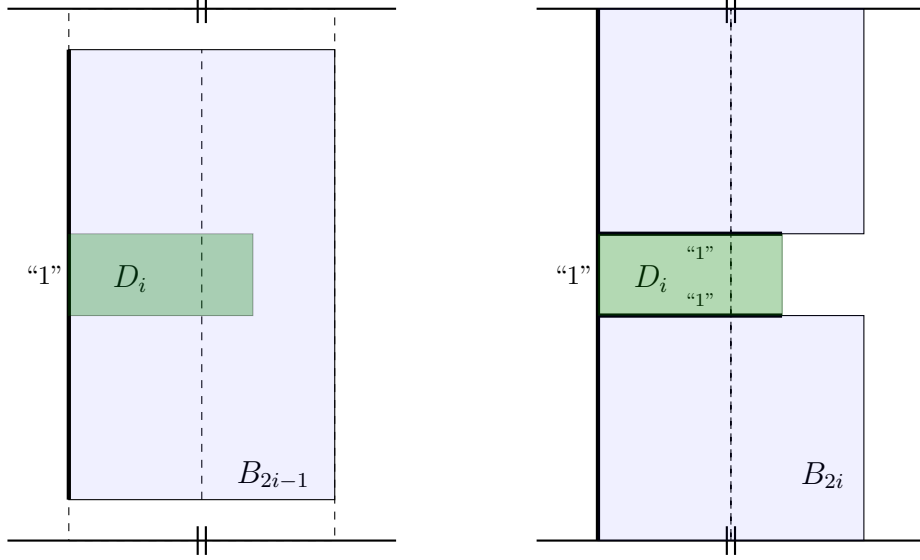


FIGURE 2. Left: the event Γ_{2i-1} where the interface of the component of $\partial_E B_{2i-1}$ does not intersect D_i . Right: the event Γ_{2i} where the interface of the component of $\partial_E B_{2i} \cup \partial_{N,S} D_i$ does not intersect the dashed line. Together, $\Gamma_{2i-1} \cap \Gamma_{2i}$ push the interface forward by ℓn .

of T , Theorem 2.6, and the bound on gap_{B_i} from (3.2) concludes the proof. Now let

$$D_i = \llbracket (i-1)\ell, (i+\frac{1}{2})\ell \rrbracket \times \llbracket \lfloor \frac{n}{2} \rfloor - \delta n, \lfloor \frac{n}{2} \rfloor + \delta n \rrbracket,$$

and define the events

$$\Gamma_{2i-1} = \left\{ X_{2i-1}^0 \upharpoonright_{B_{2i-1}} : \mathcal{I} \cap D_i = \emptyset \right\} \quad \Gamma_{2i} = \left\{ X_{2i}^0 \upharpoonright_{B_{2i}} : \mathcal{I} \cap B_{2i} - B_{2i+2} = \emptyset \right\},$$

where \mathcal{I} is the interface exposed by the cluster of $X_j^0 \upharpoonright_{\partial_W B_j}$ so that both Γ_{2i-1} and Γ_{2i} are increasing events. Then we can lower bound the right-hand side of (3.4) by

$$\mathbb{P}(A_{2i}^c \cap \Gamma_1 \cap \Gamma_2) \prod_{i=2}^N \mathbb{P}\left(A_{2i}^c \cap \Gamma_{2i-1} \cap \Gamma_{2i} \mid A_{2i-2}^c \cap \bigcap_{j \leq 2i-2} \Gamma_j\right). \quad (3.5)$$

In lower bounding the factors in (3.5), we show that there exists $c_1(\delta) > 0$ such that for all $i = 1, \dots, 2N+1$, for sufficiently large n ,

$$\mathbb{P}\left(X_{2i}^0 \upharpoonright_{\partial_W(B_{2i+1} \cup B_{2i+2})} \succeq \pi_{\mathbb{Z}^2}^1 \upharpoonright_{\partial_W(B_{2i+1} \cup B_{2i+2})} \mid \bigcap_{j \leq 2i} \Gamma_j\right) \geq 1 - ie^{-c_1 n^{\frac{1}{2} + \varepsilon}}, \quad (3.6)$$

where $X_{2i}^0 \upharpoonright_{\partial_W(B_{2i-1} \cup B_{2i})}$ is the b.c. induced on $\partial_W(B_1 \cup B_2)$ by $X_{2i}^0 \upharpoonright_{R_{2i} - B_{2i} - B_{2i-1}}$. When $i = 1$, clearly the boundary conditions on $\partial_W(B_1 \cup B_2)$ dominate $\pi_{\mathbb{Z}^2}^1 \upharpoonright_{\partial_W(B_{2i-1} \cup B_{2i})}$ since the boundary conditions there are all wired. Now fix i , suppose $A_{2i-2}^c \cap \bigcap_{j \leq 2i-2} \Gamma_j$ holds, and inductively, assume that (3.6) holds (because A_{2i-2}^c holds for $2i-2$, it also holds for X_{2i-2}^1 .) In the next step, B_{2i-1} is updated with boundary conditions dominating “wired” on $\partial_W B_{2i-1}$ and dominating all-free on $\partial_{N,S} B_{2i-1}$ and $\partial_E B_{2i-1}$.

We first bound the probability of Γ_{2i-1} and coupling on D_i as defined earlier. Let \mathcal{I}_{2i-1} be the interface revealed by the component of $\partial_w B_{2i-1}$. Observe that because Γ_{2i-1} is an increasing event, conditioned on Γ_{2i-1} , if we reveal \mathcal{I}_{2i-1} from east to west, under the monotone coupling the same edges would also be open under X_{2i-1}^1 ; then because $X_{2i-2}^0 \upharpoonright_{\partial_w B_{2i-1}} = X_{2i-2}^1 \upharpoonright_{\partial_w B_{2i-1}}$ we could couple the two chains west of that interface, and in particular on all of D_i . We claim that there exists $c(\delta) > 0$ such that

$$\begin{aligned} \mathbb{P}\left(\Gamma_{2i-1} \mid A_{2i-2}^c \cap \bigcap_{j \leq 2i-2} \Gamma_j\right) &\geq (1 - (i-1)e^{-c_1 n^{\frac{1}{2}+\varepsilon}}) \mathbb{E}_{\pi_{\mathbb{Z}^2}^1} [\pi_{B_{2i-1}}^{\xi,0}(\Gamma_{2i-1})] \\ &\gtrsim \pi_{\Lambda_{(1-2\delta)n,3\ell}}^{1,0}(\mathcal{I} \cap [\frac{n}{2} - \delta n, \frac{n}{2} + \delta n] \times [0, \frac{5\ell}{2}] = \emptyset) - e^{-cn^{\frac{1}{2}+\varepsilon}} \end{aligned}$$

where the expectation is over all boundary conditions ξ induced by $\pi_{\mathbb{Z}^2}^1$ on $\partial_E B_{2i-1}$ and $(1,0)$ boundary conditions denote wired on $\partial_S \Lambda_{(1-2\delta)n,3\ell}^{1,0}$. Indeed, the second inequality follows from considering the ℓ -enlargement $E_{\ell,2i-1}$ of B_{2i-1} which is its concentric rectangle with extra side length ℓ . If there is a wired circuit in $E_{\ell,2i-1} - B_{2i-1}$ under $\pi_{\mathbb{Z}^2}^1$ (by (2.1) this has probability $1 - e^{-c\ell}$), we can replace the expectation over b.c. induced by $\pi_{\mathbb{Z}^2}^1$ with an expectation over b.c. induced by $\pi_{E_{\ell,2i-1}}^1$. Then extending the free boundary conditions on the other three sides of B_{2i-1} all the way to $\partial_w E_{\ell,2i-1}$ and rotating yields the second inequality. By Proposition 3.5 with the choices $h = 3\ell$ and $\rho = \frac{5}{6}$, there exists $c_2(\delta, q) > 0$ so that the probability in the right-hand side above is at least $Ce^{-c_2 n^{2\varepsilon}}$ (see [9, (5.14) and Fig. 7] for a similar monotonicity argument).

Now consider the next update on B_{2i} . Under the above events, the configuration on D_i is sampled from a measure dominating $\pi_{\mathbb{Z}^2}^1$, whence by (2.1), with probability at least $1 - e^{-c\delta n}$, there is a pair of primal horizontal crossings of the top and bottom halves of D_i connecting $\partial_w B_{2i-1}$ to \mathcal{I}_{2i-1} . In that case, the boundary condition induced by X_{2i-1}^0 (also X_{2i-1}^1) on $\partial_{N,S} B_{2i} \cap D_i$ dominates “wired”. Using monotonicity in boundary conditions and (2.1) similarly to the earlier bound on Γ_{2i-1} , we obtain

$$\begin{aligned} \mathbb{P}\left(\Gamma_{2i} \mid A_{2i-2}^c \cap \Gamma_{2i-1} \cap \bigcap_{j \leq 2i-2} \Gamma_j\right) &\geq (1 - (i-1)e^{-c_1 n^{\frac{1}{2}+\varepsilon}}) \mathbb{E}_{\pi_{\mathbb{Z}^2}^1} [\pi_{B_{2i}}^{\xi,0,(i+1/2)\ell}(\Gamma_{2i})] \\ &\gtrsim \mathbb{E}_{\pi_{\mathbb{Z}^2}^1} [\pi_{\Lambda_{(1-2\delta)n,3\ell/2}}^{0,\xi,\frac{3\ell}{2}-1}(\mathcal{I} \cap \Lambda_{(1-2\delta)n,\ell})] \\ &\geq \pi_{\Lambda_{n,2\ell}}^{0,1,2\ell-1}(\mathcal{I} \cap \Lambda_{n,3\ell/2}) - e^{-cn^{\frac{1}{2}+\varepsilon}}, \end{aligned}$$

where the boundary conditions in the first line denote ξ (over which we take an expectation) induced on ∂B_{2i} east of the line $x = (i + \frac{1}{2})\ell$, and free elsewhere on ∂B_{2i} , and the boundary conditions in the second and third lines denote free on ∂_N of the boundary and respectively ξ and wired elsewhere. Here, the second inequality is a simple consequence of monotonicity in boundary conditions and the third inequality follows from again enlarging B_{2i} up to an error of $e^{-c\ell}$ coming from (2.1). By Lemma 3.4 with $\phi = 0$ and $b = 1$, there exists $c(\delta) > 0$ such that the probability on the right-hand side above is bounded below by $e^{-cn^{2\varepsilon}}$. In that case, revealing the interface from east to west, we can couple X_{2i}^0 and X_{2i}^1 beyond the interface, also implying A_{2i}^c .

Finally, we claim that under all of the above events, with probability $1 - e^{-cn^{\frac{1}{2}+\varepsilon}}$, the boundary conditions induced on $\partial_w B_{2i+1}$ and $\partial_w B_{2i+2}$ dominate those induced by $\pi_{\mathbb{Z}^2}^1$. Recall that the configuration on D_i under Γ_{2i-1} dominates $\pi_{\mathbb{Z}^2}^1|_{D_i}$. Then, since D_i contains two horizontal crossings connecting to \mathcal{I}_{2i-1} with probability $1 - 2e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$; since we are conditioning on Γ_{2i} , averaging over configurations on D_i , with probability $1 - 3e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$, in $X_{B_{2i}}^0$, $\partial_w B_{2i+2}$ is surrounded by a wired circuit. Similarly, under $X_{B_{2i}}^0$, conditional on Γ_{2i-1} and Γ_{2i} , the configuration on

$$D'_i = \llbracket i\ell, (i+2)\ell \rrbracket \times \llbracket 0, \delta n \rrbracket \cup \llbracket (1-\delta)n, n \rrbracket$$

below the interface revealed by Γ_{2i} dominates $\pi_{\mathbb{Z}^2}^1$ so that with probability $1 - 2e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$, there are horizontal primal connections to that interface in both halves of D'_i . In that case, averaging over configurations in D'_i with probability $1 - 3e^{-c\delta n^{\frac{1}{2}+\varepsilon}}$, there is a wired circuit around $\partial_w B_{2i+1}$ in $X_{B_{2i}}^0$ so that the boundary conditions induced on $\partial_w B_{2i+1}$ dominate those induced by $\pi_{\mathbb{Z}^2}^1$. Together with a union bound over the four crossing events described above, we obtain the desired for a slightly larger constant $c_1(\delta) > 0$.

Altogether this implies that for $c_2(\delta, q) > 0$,

$$\begin{aligned} \mathbb{P}\left(A_{2i}^c \cap \Gamma_{2i} \cap \Gamma_{2i-1} \mid A_{2i-2}^c \cap \bigcap_{j \leq 2i-2} \Gamma_j\right) &\gtrsim (1 - (i-1)e^{-c_1 n^{\frac{1}{2}+\varepsilon}})(1 - e^{-c_1 n^{\frac{1}{2}+\varepsilon}})e^{-c_2 n^{2\varepsilon}} \\ &\gtrsim \exp(-c_2 n^{2\varepsilon}), \end{aligned}$$

and the boundary conditions induced by $(X_k^0)_k$ on $\partial_w(B_{2i+1} \cup B_{2i+2})$ dominate that induced by $\pi_{\mathbb{Z}^2}^1$ with probability $1 - ie^{-c_1 n^{\frac{1}{2}+\varepsilon}}$. These bounds are uniform in i and therefore, plugging into (3.5) to lower bound (3.4), under the monotone coupling,

$$\mathbb{P}(X_{2N+1}^0 = X_{2N+1}^1) \gtrsim (1 - Ne^{-c_1 n^{\frac{1}{2}+\varepsilon}})(e^{-c_2 n^{2\varepsilon}})^{2N+1} \gtrsim e^{-2c_2 n^{\frac{1}{2}+\varepsilon}}. \quad \blacksquare$$

4. SLOW MIXING WITH PHASE-SYMMETRIC BOUNDARY CONDITIONS

For a reversible chain with transition kernel $P(x, y)$ and stationary distribution π , define the edge measure Q between $A, B \subset \Omega$ and conductance of the chain, Φ , by

$$Q(A, B) = \sum_{\omega \in A} \pi(\omega) \sum_{\omega' \in B} P(\omega, \omega'), \quad \Phi = \max_{\mathcal{A} \subset \Omega} \frac{Q(\mathcal{A}, \mathcal{A}^c)}{\pi(\mathcal{A})\pi(\mathcal{A}^c)}.$$

The Cheeger inequality relates these to the gap (see, e.g., [15, §7]), by stating that

$$2\Phi \geq \text{gap} \geq \Phi^2/2. \quad (4.1)$$

The torus. In [9], the authors used the above to construct an exponential bottleneck relying heavily on the topology of the torus and the exponential decay of correlations under $\pi_{\mathbb{Z}^2}^0$ at a discontinuous phase transition point. We state the result in a slightly different form, which follows from the sharp identification in [6] of the discontinuity of the phase transition for all $q > 4$.

Theorem 4.1 ([9, Theorem 3], given the result of [6]). *Let $q > 4$, and consider the critical Swendsen–Wang dynamics on $(\mathbb{Z}/n\mathbb{Z})^2$. There exists $c(q) > 0$ such that*

$$t_{\text{MIX}} \gtrsim \exp(cn).$$

For q that is sufficiently large, this was previously established in [2] (in any dimension).

Order-disorder alternating boundary conditions. Although the proof of slow mixing on the torus at a discontinuous phase transition relies heavily on the topology of the torus (see proof of Theorem 2 in [9]) we can—at least for sufficiently large q —use a similar approach to prove slow mixing in the presence of wired-free alternating boundary conditions. Exploiting the self-duality, we see that such boundary conditions still exhibit an exponential bottleneck between the ordered and disordered phases.

Definition 4.2. Let $a_n, b_n, c_n, d_n \in V(\Lambda_{n,n})$ be a set of marked vertices ordered clockwise from the origin around $\Lambda_{n,n}$ (by symmetry, without loss of generality assume $a_n \in \partial_{\text{W}}\Lambda_{n,n}$). The *alternating boundary conditions* on (a_n, b_n, c_n, d_n) are those that are red on the clockwise boundary arcs (a_n, b_n) and (c_n, d_n) and free on (b_n, c_n) and (d_n, a_n) —all connected subsets of $\partial\Lambda_{n,n}$. We say that (a_n, b_n, c_n, d_n) are ε -separated if $a_n \in \partial_{\text{W}}\Lambda_{n,n}$, at least one of $\{b_n, c_n, d_n\}$ is not contained in $\llbracket 0, \varepsilon n \rrbracket \times \llbracket 0, n \rrbracket$, and

$$\min_{i,j \in a,b,c,d; i \neq j} \|i_n - j_n\|_{\infty} \geq \varepsilon n.$$

Remark 4.3. The requirement of ε -separation in our consideration of alternating boundary conditions arises from the fact that if (a_n, b_n, c_n, d_n) were all on $\partial_{\text{W}}\Lambda_{n,n}$ repeating the proof of [9, Eq. (1.4) of Theorem 3] with such boundary conditions would yield that the mixing time is in fact sub-exponential. Clearly, if the four marked vertices are sufficiently close to being on one side or to each other, a similar picture would emerge. The requirement of macroscopic separation ensures phase symmetry.

With Definition 4.2 in hand, the following is a precise reformulation of Theorem 1.

Proposition 4.4. *Let q be large, $\varepsilon > 0$, and consider the Swendsen–Wang dynamics for the critical Potts model on $\Lambda = \Lambda_{n,n}$ with alternating boundary (alt) on (a_n, b_n, c_n, d_n) that are ε -separated. Then there exists $c(q) > 0$ such that*

$$t_{\text{MIX}}^{\text{alt}} \gtrsim \exp(c\varepsilon n).$$

Proof of Theorem 1. By (2.6) and Fact 2.10, it suffices to prove the bound for the FK Glauber dynamics with alternating FK boundary conditions that are wired on the boundary arcs (a_n, b_n) and $(c_n, d_n) \subset \partial\Lambda_{n,n}$ and free elsewhere. Observe that

$$\{(a_n, b_n) \longleftrightarrow (c_n, d_n)\} = \{(b_n, c_n) \xleftrightarrow{*} (d_n, a_n)\}^c,$$

and therefore, either

$$\pi_{\Lambda}^{\text{alt}}((a_n, b_n) \longleftrightarrow (c_n, d_n)) \leq \frac{1}{2}, \quad \text{or} \quad \pi_{\Lambda}^{\text{alt}}((b_n, c_n) \xleftrightarrow{*} (d_n, a_n)) \leq \frac{1}{2}.$$

By self-duality of the class of ε -separated, alternating boundary conditions, we can suppose without loss of generality that we are in the former case.

Recall the definition of the strips $\mathcal{S}_{b,h,\phi}$ in Definition 3.1, and let $a_n = (a_n^1, a_n^2)$, and likewise for b_n, c_n, d_n . Let $\phi_{a,d} = \tan^{-1}(\frac{d_n^2 - a_n^2}{d_n^1 - a_n^1})$ and $\phi_{b,c} = \tan^{-1}(\frac{c_n^2 - b_n^2}{c_n^1 - b_n^1})$. Observe that

by the ε -separation, one of $\phi_{a,d}$ and $\phi_{b,c}$ is in $[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta]$ for some small enough δ depending only on ε . Suppose without loss of generality that $\phi_{a,d}$ is and consider strips

$$\begin{aligned} S_1 &= \mathcal{H}_{a_n^1, \phi_{a,d}}^+ \cap \mathcal{H}_{a_n^1 + \|b_n - a_n\|_\infty, \phi_{a,d}}^- \cap \Lambda, \\ S_2 &= \mathcal{H}_{a_n^1, \phi_{a,d}}^+ \cap \mathcal{H}_{a_n^1 + \|c_n - d_n\|_\infty, \phi_{a,d}}^- \cap \Lambda. \end{aligned}$$

Geometrically, one of S_1, S_2 must satisfy $S_i \cap \partial\Lambda \subset (a_n, b_n) \cup (c_n, d_n)$. Let S be that strip S_i ; because $S = S_i$ for some $i = 1, 2$, there is some x, h such that $S = \mathcal{S}_{x,h,\phi_{a,d}} \cap \Lambda$; fix that $x \in \mathbb{R}_+$, $h \geq \frac{\varepsilon}{2}$. Define $\partial_N S = S \cap \mathcal{H}_{x+h-1,\phi}^+$ and $\partial_S S = S \cap \mathcal{H}_{x-h+1,\phi}^-$, and let

$$\mathcal{A} = \left\{ (a_n, b_n) \xleftrightarrow{S} (c_n, d_n) \right\}$$

be the bottleneck set whose conductance $Q(\mathcal{A}, \mathcal{A}^c)$ we bound. Note that since

$$\mathcal{A} \subset \{ (a_n, b_n) \longleftrightarrow (c_n, d_n) \},$$

we have that $\pi_\Lambda^{\text{alt}}(\mathcal{A}^c) > \frac{1}{2}$. Therefore, it suffices to prove that for some $c(q) > 0$,

$$\pi_\Lambda^{\text{alt}}(\partial\mathcal{A} \mid \mathcal{A}) \lesssim \exp(-c\varepsilon n),$$

(where $\partial\mathcal{A} := \{\omega : P(\omega, \mathcal{A}^c) > 0\}$) in which case,

$$\text{gap} \leq 2\Phi \leq \frac{2Q(\mathcal{A}, \mathcal{A}^c)}{\pi_\Lambda^{\text{alt}}(\mathcal{A})\pi_\Lambda^{\text{alt}}(\mathcal{A}^c)} \leq 4\pi_\Lambda^{\text{alt}}(\partial\mathcal{A} \mid \mathcal{A}),$$

(where the last inequality follows from using a worst case bound of 1 on the transition rates in $Q(\mathcal{A}, \mathcal{A}^c)/\pi(\mathcal{A})$), implies the desired lower bound on gap^{-1} .

For $\omega \in \mathcal{A}$, in order for $P(\omega, \mathcal{A}^c)$ to be positive ($\omega \in \partial\mathcal{A}$), there must exist an edge e in \mathcal{S} that is *pivotal* to \mathcal{A} , i.e., $\omega(e) = 1$ and $\omega' = \omega - \{e\} \notin \mathcal{A}$. We estimate the probability $\pi_\Lambda^{\text{alt}}(\partial\mathcal{A} \mid \mathcal{A})$ by union bounding over the probability of any edge, e , in $E(S)$ being pivotal to \mathcal{A} .

First examine whether e is closer in $\|\cdot\|_\infty$ to $\partial_N S$ or $\partial_S S$. Suppose without loss of generality, we are in the former case, whence we expose the north-most primal crossing of S , under $\pi_\Lambda^{\text{alt}}(\omega \mid \mathcal{A})$ (revealing, first, the configuration on $\Lambda \cap \mathcal{H}_{x+h-1,\phi}^+$, then the dual-components of $\partial_N S$ in S). Such a crossing exists by the conditioning on \mathcal{A} .

Denote by ζ the horizontal crossing we have revealed as such. By the conditioning on S , it is clear that ζ must connect (a_n, b_n) to (c_n, d_n) in S . In order for e to be pivotal to S , e must be an open edge in ζ and there must exist a dual crossing connecting e to $\partial_S S$. Let D be the southern connected component of $\Lambda - \zeta$; we wish to bound

$$\pi_\Lambda^{\text{alt}} \left(e \xleftrightarrow{D^*} \partial_S S \mid \mathcal{A}, \omega \upharpoonright_\zeta = 1 \right).$$

By monotonicity in boundary conditions, if we let $R = D \cup S$,

$$\pi_\Lambda^{\text{alt}}(\omega \upharpoonright_D \mid \mathcal{A}, \omega \upharpoonright_\zeta = 1) = \pi_D^{1,0} \succeq \pi_R^{1,0}(\omega \upharpoonright_D),$$

where $(1, 0)$ boundary conditions denote free on $\partial R - S$ and wired elsewhere.

We can decompose the probability

$$\pi_R^{1,0} \left(e \xleftrightarrow{D^*} \partial_S S \right) \leq \pi_R^{1,0} \left(e \xleftrightarrow{*} \partial_S S \right)$$

into the event Γ_1 that the dual-component of $\partial_s S$ (and thus the interface of R with $(1, 0)$ boundary conditions) is not a subset of $S \cap \mathcal{H}_{x-h/2, \phi}^+$, and Γ_1^c . Under Γ_1^c , since e is closer in ℓ_∞ to $\partial_N S$, $d_\infty(e, \partial_s S) \geq h/2$ so that $e \notin S \cap \mathcal{H}_{x-h/2, \phi}^-$ and e cannot be dual-connected to $\partial_s S$.

Bounding the probability of Γ_1^c , by Lemma 3.4, there exists $c(\varepsilon, q) > 0$ such that

$$\pi_\Lambda^{\text{alt}}(\Gamma_1 \mid \mathcal{A}, \omega|_\zeta = 1) \leq \pi_R^{(1,0)}(\Gamma_1) \lesssim \exp(-c\varepsilon n).$$

Under Γ_1 we can take a worst case bound of one on the probability of $e \xleftrightarrow{D^*} \partial_s S$. Therefore, for some $c(q) > 0$, we have $\pi_\Lambda^{\text{alt}}(\partial \mathcal{A} \mid \mathcal{A}) \lesssim \exp(-c\varepsilon n)$.

Using the above as a bound on $Q(\mathcal{A}, \mathcal{A}^c)/\pi_\Lambda^{\text{alt}}(\mathcal{A})$ in Eq. (4.1) and plugging into Eq. (4.1) implies for the FK Glauber dynamics and, by Eq. (2.6), Swendsen–Wang dynamics with alternating order-disorder boundary conditions, $\text{gap}^{-1} \gtrsim \exp(c\varepsilon n)$. ■

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